Lesson 16 and 17: Oscillations

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1 Learning goals for this lesson

The following is a list of basic ideas that were covered during this lesson. You must be fairly conversant with all of them now. If you have any questions, please post them on Piazza. The link is `piazza.com/phys211`.

- What is an oscillation and why is it important to learn about these?
- How do we describe oscillatory motion?
- Understand a simple example of a real system that produces simple harmonic oscillation - springs.
- Recognize that the pendulum also executes simples harmonic motion for small oscillations and understand it.

2 A quick summary

We now move on to one of the most important topics in all of physics - oscillations. A lot of what we experience in day to day life is a result of oscillations – sound, light, cell phone signals, TV. We rely on sound waves to communicate with each other - sound waves are nothing but a back and forth motion of air molecules. Oscillations of the vocal cords makes the air around oscillate, and our ears pick this up via the ear drum and the brain recognizes tiny changes in pressure of the fluid in the inner ear due to this, and we hear. Oscillations are the basis of light and sight.

Trying to understand generic oscillations is a fairly complex question, and like all other things, we start with trying to understand the simplest version of it. We will start with a system that we are familiar with, and see how it can also describe oscillatory motion.

Consider a particle moving in a circle in the xy plane.. If it is moving at constant angular velocity ω, then, the particles x coordinate is given by $x = R\cos\theta$, where R is the radius of the circle. At constant angular velocity, $\theta(t) = \theta_0 + \omega t$. Thus, the x coordinate of the particle is $x = R\cos(\theta_0 + \omega t)$, which describes an oscillating function. In fact sine and cosine are the simplest oscillatory functions we can think about. They move back and forth regularly at a constant rate. Interestingly, it is enough to understand these simple oscillations, as all oscillations, even very complex ones are constructed from sines and cosines.

We will now forget that we started out with circular motion and consider the motion of an object described by

$$x(t) = A\sin(\omega t + \phi). \tag{1}$$

Motion described by the above equation for the positon as a function of time is called "simple harmonic motion". It is the simplest form of oscillatory behavior. We have changed notation a bit to conform to standards. We have changed the radius R to the "amplitude" A, which is the maximum absolute value that x can take. We have changed the sine to a cosine since they are related functions ($\sin(\pi/2+\theta) = \cos\theta$). The starting angle θ_0 has been replaced with the "phase" ϕ, which also describes the starting position (i.e., at time $t = 0$) of the particle. At $t = 0$, the particle has a position $x(0) = A\cos\phi$. The velocity of the particle is

$$v(t) = \frac{dx}{dt} = A\omega\cos(\omega t + \phi). \tag{2}$$

The maximum value of the magnitude of the velocity is $A\omega$. This value is reached when the argument of the cosine is 0 or π, since $\cos 0 = 1$ and $\cos\pi = -1$, i.e, $\omega t + \phi = \pi, 2\pi, 3\pi$, etc.. At this time, the position is $x = 0$, since $\sin(\pi) = 0$, etc. The velocity is therefore maximum when the particle is at the origin. This makes sense – as the particle reaches the extremities of the motion it's velocity hits 0, and the position is maximized. As it returns to the origin, it speeds up, and once it passes the origin, it slows down again.

We now proceed to find the acceleration of the particle as a function of time.

$$a(t) = \frac{dv}{dt} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x(t). \tag{3}$$

We see that the acceleration of the particle is proportional to the displacement and in a direction opposing it. This is really the basis of all oscillations – an acceleration that is trying to oppose the motion of the particle. So if the particle starts out at some positive value of x, then the acceleration points in the negative direction. This produces a negative velocity, and the particle starts moving towards the origin. As is approaches the origin, it has some velocity due to the accleration and it overshoots the origin. It now has a negative position, and the acceleration points in the positive directon, thus send the particle back towards the origin. If there is no loss of energy via friction, this motion carries on forever. ω is a remnant of our analogy with circular motion. It is called the "angular frequency" of the oscillation. We can equivalently describe a "frequency" as the no. of oscillations per second. This is defined as $f = \omega/2\pi$. The time period of the oscillation is the amount of time taken for one complete oscillation, and is given by $T = 1/f = 2\pi/\omega$. The units of frequency are oscillations per second or cycles per second, also known as "Hertz". Time period, of course, is measure in seconds in standard units.

In general, for any system, if there is some "restoring force", i.e., some force that tends to oppose the motion, then oscillations will occur. If the accleration is proportional to the displacement and opposite to it (minus sign), then the proportionality constant is the square of the angular frequency of the oscillation. Let us consider an example – the simple pendulum.

A simple pendulum consists of a "bob" which is basically a mass hanging at the end of a massless string. It is common knowledge that if one kicks a pendulum, it oscillates back and forth. We would like to understand this oscillation and if possible quantify it. Figure 1 shows a pendulum displaced from the vertical. The only forces on the pendulum are the tension in the string and gravitational force. Gravity always acts vertically downwards, while tension always acts along the string. Since the string's length is fixed, the tension balances out any force in the radial direction, and the force component in the tangential direction (from gravity) makes the pendulum return. Writing Newtons equations of motion for the direction tangential to the motion, we get,

$$F_t = -mg\sin\theta = ma_t \tag{4}$$

The negative sign exists because the acceleration is to the right, while the displacment is to the left. In other words, the force is a restoring force. Since the tangential acceleration is related to the angular acceleration, $a_t = l\alpha$, l being the lenght of the string, we get,

$$ml\alpha = -mg\sin\theta. \tag{5}$$

This is a difficult equation to solve in general, but we can make an approximation. For very small angles, one can check that, $\sin\theta \approx \theta$. θ of course is in radians. This approximation is reasonable when $\theta < 0.1$. With this, we get,

$$\alpha = -\frac{g}{l}\theta. \tag{6}$$

What this is telling us is that the angular acceleration is proprtional to the angular displacment, and is opposite to it. This is exactly the equation we had for simple harmonic motion above, except there it was for linear motion, not angular, but the story is the same. The proportionality constant, we recognize, by comparison as the square of the angular frequency.

$$\omega^2 = \frac{g}{l}. \tag{7}$$

The time period is therefore

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{g}{l}}. \tag{8}$$

This is the oscillation period of a simple pendulum for small oscillations. Note that it doesn't depend on the mass of the pendulum bob. Measuring the length and the time period of a physical pendulum will allow us to get an estimate for the gravitational acceleration g.

Now that we understand how to find the oscillation frequency and period, we will discuss another oscillation system. The pendulum oscillated under gravity – gravity provided the restoring force. Most materials have some inherent elasticity. We now talk explicitly about "springs" Springs resist both compression and expansion. A simple model for the spring says that the force applied by the spring is directly proportional to the compression or expansion. If Δx is the expansion of the spring from its natural length (a negative value would correspond to compression), then the restoring force, or the force trying to bring the spring back to its natural form is given by,

$$F = -k\Delta x. \tag{9}$$

The constant of proportionality k is called the spring constant, and the above model for a spring is called the Hooke's law. It is the simplest model for a spring, or for elastic materials. If a mass m is attached at the end of such a spring and stretched a distance Δx, the the acceleration of the mass is given by

$$a = \frac{F}{m} = -\frac{k}{m}\Delta x. \tag{10}$$

We again have a situation where the acceleration of the object is proportional to the displacement and in the opposite direction. This tells us that the resulting motion is simple harmonic with an angular frequency $\omega = \sqrt{k/m}$. The time period, $T = 2\pi\sqrt{m/k}$.

We have thus described oscillatory motion, seen that it can be represented in terms of sines and cosines, and that it is equivalent to circular motion. We have also seen two examples of systems that exhibit oscillatory behaviour, one, the pendulum, under gravity, and another, the spring, which is governed by the Hooke's law.

We would like calculate one more quanitity, the work done in compression a spring by a certain amount. Since a compressed spring can exert a force, and wants to expand, it has the potential to produce motion. We therefore say that a compressed spring has some potential energy. In order to find what it is, we calculate the work done in compressing a spring by a certain amount.

Let's say that a spring is at its natural length. We would like to find out the work done to compress it by an amount Δx. At some point during the compression, let's say at a compression of x, the spring force is

$$F = -kx, \tag{11}$$

so the force that we are applying to keep it compressed is opposite to the spring force and is equal to kx. In compressing it a further distance dx, we do work on it that is given by,

$$dW = F_{\text{on spring}} dx = kx\, dx. \tag{12}$$

The total work done in compressing the spring from the natural length to Δx is

$$W = \int_0^{\Delta x} kx\, dx = \frac{1}{2}k(\Delta x)^2. \tag{13}$$

Thus, in compressing a spring by Δx, we store an energy equal to $\frac{1}{2}k(\Delta x)^2$ in it. This is the spring potential energy, and it can be converted into kinetic energy e.g., when a spring is allowed to act on an object.

3 Additional resources

- PhET demo on springs - look at `http://phet.colorado.edu`
- Read the textbook sections on oscillations - we have not covered damped and force oscillations.

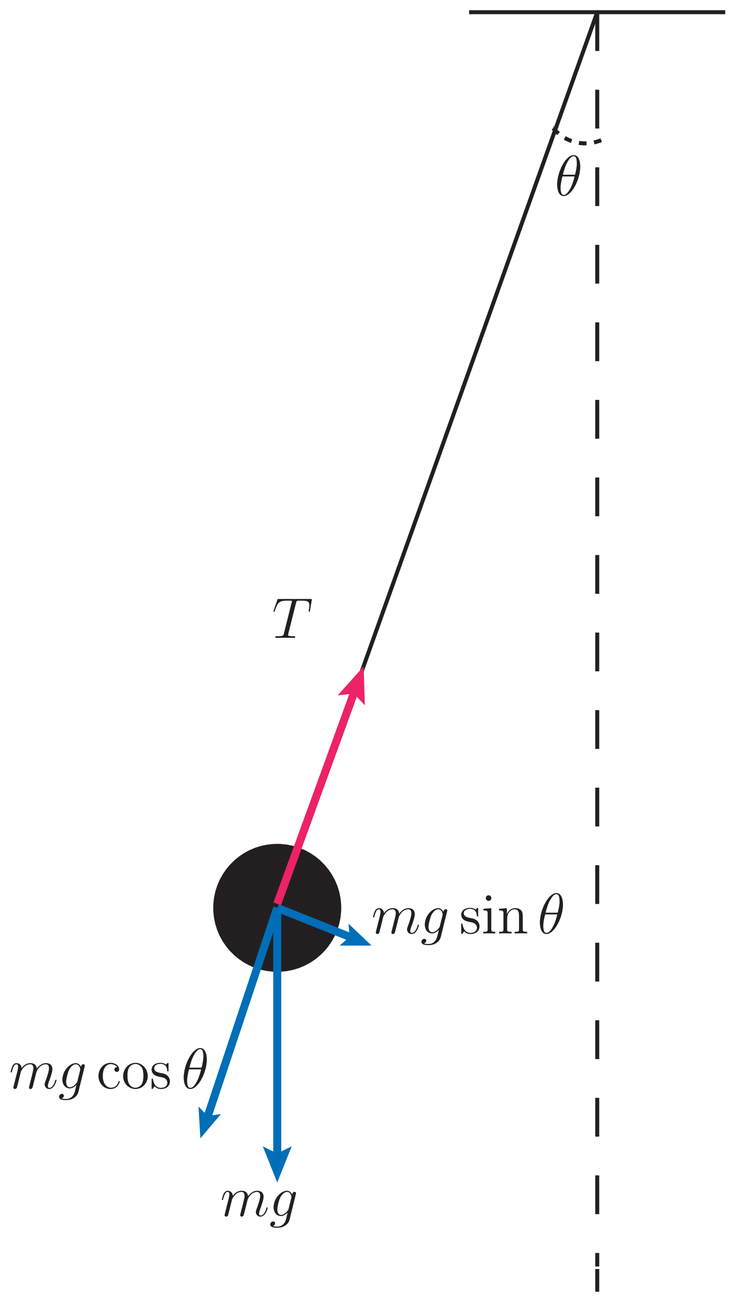

Figure 1: Simple pendulum