***3.1-3***
Explain why the statement, "The running time of algorithm $A$ is at least $O(n^2)$," is meaningless.

***3.1-4***
Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

***3.1-5***
Prove Theorem 3.1.

***3.1-6***
Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

***3.1-7***
Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

***3.1-8***
We can extend our notation to the case of two parameters $n$ and $m$ that can go to infinity independently at different rates. For a given function $g(n, m)$, we denote by $O(g(n, m))$ the set of functions

$$\begin{aligned} O(g(n,m)) = \{f(n,m) : \ & \text{there exist positive constants } c,\ n_0, \text{ and } m_0 \\ & \text{such that } 0 \le f(n,m) \le cg(n,m) \\ & \text{for all } n \ge n_0 \text{ or } m \ge m_0\} \ . \end{aligned}$$

Give corresponding definitions for $\Omega(g(n, m))$ and $\Theta(g(n, m))$.

## 3.2 Standard notations and common functions

This section reviews some standard mathematical functions and notations and explores the relationships among them. It also illustrates the use of the asymptotic notations.

### Monotonicity

A function $f(n)$ is ***monotonically increasing*** if $m \le n$ implies $f(m) \le f(n)$. Similarly, it is ***monotonically decreasing*** if $m \le n$ implies $f(m) \ge f(n)$. A function $f(n)$ is ***strictly increasing*** if $m < n$ implies $f(m) < f(n)$ and ***strictly decreasing*** if $m < n$ implies $f(m) > f(n)$.

## Floors and ceilings

For any real number $x$, we denote the greatest integer less than or equal to $x$ by $\lfloor x \rfloor$ (read "the floor of $x$") and the least integer greater than or equal to $x$ by $\lceil x \rceil$ (read "the ceiling of $x$"). For all real $x$,

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1 . \tag{3.3}$$

For any integer $n$,

$$\lceil n/2 \rceil + \lfloor n/2 \rfloor = n ,$$

and for any real number $x \ge 0$ and integers $a, b > 0$,

$$\left\lceil \frac{\lceil x/a \rceil}{b} \right\rceil = \left\lceil \frac{x}{ab} \right\rceil , \tag{3.4}$$

$$\left\lfloor \frac{\lfloor x/a \rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor , \tag{3.5}$$

$$\left\lceil \frac{a}{b} \right\rceil \le \frac{a + (b-1)}{b} , \tag{3.6}$$

$$\left\lfloor \frac{a}{b} \right\rfloor \ge \frac{a - (b-1)}{b} . \tag{3.7}$$

The floor function $f(x) = \lfloor x \rfloor$ is monotonically increasing, as is the ceiling function $f(x) = \lceil x \rceil$.

## Modular arithmetic

For any integer $a$ and any positive integer $n$, the value $a \bmod n$ is the ***remainder*** (or ***residue***) of the quotient $a/n$:

$$a \bmod n = a - n \lfloor a/n \rfloor . \tag{3.8}$$

It follows that

$$0 \le a \bmod n < n . \tag{3.9}$$

Given a well-defined notion of the remainder of one integer when divided by another, it is convenient to provide special notation to indicate equality of remainders. If $(a \bmod n) = (b \bmod n)$, we write $a \equiv b \pmod{n}$ and say that $a$ is ***equivalent*** to $b$, modulo $n$. In other words, $a \equiv b \pmod{n}$ if $a$ and $b$ have the same remainder when divided by $n$. Equivalently, $a \equiv b \pmod{n}$ if and only if $n$ is a divisor of $b - a$. We write $a \not\equiv b \pmod{n}$ if $a$ is not equivalent to $b$, modulo $n$.

### Polynomials

Given a nonnegative integer $d$, a ***polynomial in n of degree d*** is a function $p(n)$ of the form

$$p(n) = \sum_{i=0}^{d} a_i n^i ,$$

where the constants $a_0, a_1, \ldots, a_d$ are the ***coefficients*** of the polynomial and $a_d \neq 0$. A polynomial is asymptotically positive if and only if $a_d > 0$. For an asymptotically positive polynomial $p(n)$ of degree $d$, we have $p(n) = \Theta(n^d)$. For any real constant $a \geq 0$, the function $n^a$ is monotonically increasing, and for any real constant $a \leq 0$, the function $n^a$ is monotonically decreasing. We say that a function $f(n)$ is ***polynomially bounded*** if $f(n) = O(n^k)$ for some constant $k$.

### Exponentials

For all real $a > 0$, $m$, and $n$, we have the following identities:

$$\begin{aligned} a^0 &= 1 , \\ a^1 &= a , \\ a^{-1} &= 1/a , \\ (a^m)^n &= a^{mn} , \\ (a^m)^n &= (a^n)^m , \\ a^m a^n &= a^{m+n} . \end{aligned}$$

For all $n$ and $a \geq 1$, the function $a^n$ is monotonically increasing in $n$. When convenient, we shall assume $0^0 = 1$.

We can relate the rates of growth of polynomials and exponentials by the following fact. For all real constants $a$ and $b$ such that $a > 1$,

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 , \tag{3.10}$$

from which we can conclude that

$$n^b = o(a^n) .$$

Thus, any exponential function with a base strictly greater than 1 grows faster than any polynomial function.

Using $e$ to denote $2.71828\ldots$, the base of the natural logarithm function, we have for all real $x$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!} , \tag{3.11}$$

where "!" denotes the factorial function defined later in this section. For all real $x$, we have the inequality

$$e^x \geq 1 + x , \tag{3.12}$$

where equality holds only when $x = 0$. When $|x| \leq 1$, we have the approximation

$$1 + x \leq e^x \leq 1 + x + x^2 . \tag{3.13}$$

When $x \to 0$, the approximation of $e^x$ by $1 + x$ is quite good:

$$e^x = 1 + x + \Theta(x^2) .$$

(In this equation, the asymptotic notation is used to describe the limiting behavior as $x \to 0$ rather than as $x \to \infty$.) We have for all $x$,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x . \tag{3.14}$$

## Logarithms

We shall use the following notations:

$$\begin{aligned} \lg n &= \log_2 n && \text{(binary logarithm)} , \\ \ln n &= \log_e n && \text{(natural logarithm)} , \\ \lg^k n &= (\lg n)^k && \text{(exponentiation)} , \\ \lg\lg n &= \lg(\lg n) && \text{(composition)} . \end{aligned}$$

An important notational convention we shall adopt is that *logarithm functions will apply only to the next term in the formula*, so that $\lg n + k$ will mean $(\lg n) + k$ and not $\lg(n + k)$. If we hold $b > 1$ constant, then for $n > 0$, the function $\log_b n$ is strictly increasing.

For all real $a > 0$, $b > 0$, $c > 0$, and $n$,

$$\begin{aligned} a &= b^{\log_b a} , \\ \log_c(ab) &= \log_c a + \log_c b , \\ \log_b a^n &= n \log_b a , \\ \log_b a &= \frac{\log_c a}{\log_c b} , && (3.15) \\ \log_b(1/a) &= -\log_b a , \\ \log_b a &= \frac{1}{\log_a b} , \\ a^{\log_b c} &= c^{\log_b a} , && (3.16) \end{aligned}$$

where, in each equation above, logarithm bases are not 1.

By equation (3.15), changing the base of a logarithm from one constant to another changes the value of the logarithm by only a constant factor, and so we shall often use the notation "$\lg n$" when we don't care about constant factors, such as in $O$-notation. Computer scientists find 2 to be the most natural base for logarithms because so many algorithms and data structures involve splitting a problem into two parts.

There is a simple series expansion for $\ln(1 + x)$ when $|x| < 1$:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

We also have the following inequalities for $x > -1$:

$$\frac{x}{1 + x} \le \ln(1 + x) \le x , \tag{3.17}$$

where equality holds only for $x = 0$.

We say that a function $f(n)$ is ***polylogarithmically bounded*** if $f(n) = O(\lg^k n)$ for some constant $k$. We can relate the growth of polynomials and polylogarithms by substituting $\lg n$ for $n$ and $2^a$ for $a$ in equation (3.10), yielding

$$\lim_{n \to \infty} \frac{\lg^b n}{(2^a)^{\lg n}} = \lim_{n \to \infty} \frac{\lg^b n}{n^a} = 0 .$$

From this limit, we can conclude that

$$\lg^b n = o(n^a)$$

for any constant $a > 0$. Thus, any positive polynomial function grows faster than any polylogarithmic function.

## Factorials

The notation $n!$ (read "$n$ factorial") is defined for integers $n \ge 0$ as

$$n! = \begin{cases} 1 & \text{if } n = 0 , \\ n \cdot (n - 1)! & \text{if } n > 0 . \end{cases}$$

Thus, $n! = 1 \cdot 2 \cdot 3 \cdots n$.

A weak upper bound on the factorial function is $n! \le n^n$, since each of the $n$ terms in the factorial product is at most $n$. ***Stirling's approximation***,

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) , \tag{3.18}$$

where $e$ is the base of the natural logarithm, gives us a tighter upper bound, and a lower bound as well. As Exercise 3.2-3 asks you to prove,

$$\begin{aligned} n! &= o(n^n) , \\ n! &= \omega(2^n) , \\ \lg(n!) &= \Theta(n \lg n) , \end{aligned} \tag{3.19}$$

where Stirling's approximation is helpful in proving equation (3.19). The following equation also holds for all $n \geq 1$:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n} \tag{3.20}$$

where

$$\frac{1}{12n+1} < \alpha_n < \frac{1}{12n} . \tag{3.21}$$

## Functional iteration

We use the notation $f^{(i)}(n)$ to denote the function $f(n)$ iteratively applied $i$ times to an initial value of $n$. Formally, let $f(n)$ be a function over the reals. For non-negative integers $i$, we recursively define

$$f^{(i)}(n) = \begin{cases} n & \text{if } i = 0 , \\ f(f^{(i-1)}(n)) & \text{if } i > 0 . \end{cases}$$

For example, if $f(n) = 2n$, then $f^{(i)}(n) = 2^i n$.

## The iterated logarithm function

We use the notation $\lg^* n$ (read "log star of $n$") to denote the iterated logarithm, defined as follows. Let $\lg^{(i)} n$ be as defined above, with $f(n) = \lg n$. Because the logarithm of a nonpositive number is undefined, $\lg^{(i)} n$ is defined only if $\lg^{(i-1)} n > 0$. Be sure to distinguish $\lg^{(i)} n$ (the logarithm function applied $i$ times in succession, starting with argument $n$) from $\lg^i n$ (the logarithm of $n$ raised to the $i$th power). Then we define the iterated logarithm function as

$$\lg^* n = \min\left\{i \geq 0 : \lg^{(i)} n \leq 1\right\} .$$

The iterated logarithm is a *very* slowly growing function:

$$\begin{aligned} \lg^* 2 &= 1 , \\ \lg^* 4 &= 2 , \\ \lg^* 16 &= 3 , \\ \lg^* 65536 &= 4 , \\ \lg^*(2^{65536}) &= 5 . \end{aligned}$$

Since the number of atoms in the observable universe is estimated to be about $10^{80}$, which is much less than $2^{65536}$, we rarely encounter an input size $n$ such that $\lg^* n > 5$.

## Fibonacci numbers

We define the ***Fibonacci numbers*** by the following recurrence:

$$\begin{aligned} F_0 &= 0 , \\ F_1 &= 1 , \\ F_i &= F_{i-1} + F_{i-2} \qquad \text{for } i \geq 2 . \end{aligned} \tag{3.22}$$

Thus, each Fibonacci number is the sum of the two previous ones, yielding the sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... .

Fibonacci numbers are related to the ***golden ratio*** $\phi$ and to its conjugate $\hat{\phi}$, which are the two roots of the equation

$$x^2 = x + 1 \tag{3.23}$$

and are given by the following formulas (see Exercise 3.2-6):

$$\begin{aligned} \phi &= \frac{1+\sqrt{5}}{2} \\ &= 1.61803\ldots , \\ \hat{\phi} &= \frac{1-\sqrt{5}}{2} \\ &= -.61803\ldots . \end{aligned} \tag{3.24}$$

Specifically, we have

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} ,$$

which we can prove by induction (Exercise 3.2-7). Since $|\hat{\phi}| < 1$, we have

$$\begin{aligned} \frac{|\hat{\phi}^i|}{\sqrt{5}} &< \frac{1}{\sqrt{5}} \\ &< \frac{1}{2} , \end{aligned}$$

which implies that

$$F_i = \left\lfloor \frac{\phi^i}{\sqrt{5}} + \frac{1}{2} \right\rfloor , \qquad (3.25)$$

which is to say that the $i$th Fibonacci number $F_i$ is equal to $\phi^i/\sqrt{5}$ rounded to the nearest integer. Thus, Fibonacci numbers grow exponentially.

## Exercises

***3.2-1***
Show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so are the functions $f(n) + g(n)$ and $f(g(n))$, and if $f(n)$ and $g(n)$ are in addition nonnegative, then $f(n) \cdot g(n)$ is monotonically increasing.

***3.2-2***
Prove equation (3.16).

***3.2-3***
Prove equation (3.19). Also prove that $n! = \omega(2^n)$ and $n! = o(n^n)$.

***3.2-4*** ★
Is the function $\lceil \lg n \rceil!$ polynomially bounded? Is the function $\lceil \lg \lg n \rceil!$ polynomially bounded?

***3.2-5*** ★
Which is asymptotically larger: $\lg(\lg^* n)$ or $\lg^*(\lg n)$?

***3.2-6***
Show that the golden ratio $\phi$ and its conjugate $\hat{\phi}$ both satisfy the equation $x^2 = x + 1$.

***3.2-7***
Prove by induction that the $i$th Fibonacci number satisfies the equality

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} ,$$

where $\phi$ is the golden ratio and $\hat{\phi}$ is its conjugate.

***3.2-8***
Show that $k \ln k = \Theta(n)$ implies $k = \Theta(n/\ln n)$.