A Summations

When an algorithm contains an iterative control construct such as a **while** or **for** loop, we can express its running time as the sum of the times spent on each execution of the body of the loop. For example, we found in Section 2.2 that the jth iteration of insertion sort took time proportional to j in the worst case. By adding up the time spent on each iteration, we obtained the summation (or series)

$$\sum_{j=2}^{n} j .$$

When we evaluated this summation, we attained a bound of $\Theta(n^2)$ on the worst-case running time of the algorithm. This example illustrates why you should know how to manipulate and bound summations.

Section A.1 lists several basic formulas involving summations. Section A.2 offers useful techniques for bounding summations. We present the formulas in Section A.1 without proof, though proofs for some of them appear in Section A.2 to illustrate the methods of that section. You can find most of the other proofs in any calculus text.

A.1 Summation formulas and properties

Given a sequence $a_1, a_2, \ldots, a_n$ of numbers, where n is a nonnegative integer, we can write the finite sum $a_1 + a_2 + \cdots + a_n$ as

$$\sum_{k=1}^{n} a_k .$$

If $n = 0$, the value of the summation is defined to be 0. The value of a finite series is always well defined, and we can add its terms in any order.

Given an infinite sequence $a_1, a_2, \ldots$ of numbers, we can write the infinite sum $a_1 + a_2 + \cdots$ as

$$\sum_{k=1}^{\infty} a_k ,$$

which we interpret to mean

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k .$$

If the limit does not exist, the series ***diverges***; otherwise, it ***converges***. The terms of a convergent series cannot always be added in any order. We can, however, rearrange the terms of an ***absolutely convergent series***, that is, a series $\sum_{k=1}^{\infty} a_k$ for which the series $\sum_{k=1}^{\infty} |a_k|$ also converges.

Linearity

For any real number c and any finite sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\sum_{k=1}^{n} (ca_k + b_k) = c\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k .$$

The linearity property also applies to infinite convergent series.

We can exploit the linearity property to manipulate summations incorporating asymptotic notation. For example,

$$\sum_{k=1}^{n} \Theta(f(k)) = \Theta\left(\sum_{k=1}^{n} f(k)\right) .$$

In this equation, the Θ-notation on the left-hand side applies to the variable k, but on the right-hand side, it applies to n. We can also apply such manipulations to infinite convergent series.

Arithmetic series

The summation

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n ,$$

is an ***arithmetic series*** and has the value

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) \tag{A.1}$$

$$= \Theta(n^2) . \tag{A.2}$$

Sums of squares and cubes

We have the following summations of squares and cubes:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{A.3}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \tag{A.4}$$

Geometric series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

is a ***geometric*** or ***exponential series*** and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}. \tag{A.5}$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}. \tag{A.6}$$

Harmonic series

For positive integers n, the nth ***harmonic number*** is

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{1}{k} \\ &= \ln n + O(1). \end{aligned} \tag{A.7}$$

(We shall prove a related bound in Section A.2.)

Integrating and differentiating series

By integrating or differentiating the formulas above, additional formulas arise. For example, by differentiating both sides of the infinite geometric series (A.6) and multiplying by x, we get

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \tag{A.8}$$

for $|x| < 1$.

Telescoping series

For any sequence $a_0, a_1, \ldots, a_n$,

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0 , \tag{A.9}$$

since each of the terms $a_1, a_2, \ldots, a_{n-1}$ is added in exactly once and subtracted out exactly once. We say that the sum ***telescopes***. Similarly,

$$\sum_{k=0}^{n-1} (a_k - a_{k+1}) = a_0 - a_n .$$

As an example of a telescoping sum, consider the series

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} .$$

Since we can rewrite each term as

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} ,$$

we get

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{n} . \end{aligned}$$

Products

We can write the finite product $a_1 a_2 \cdots a_n$ as

$$\prod_{k=1}^{n} a_k .$$

If $n = 0$, the value of the product is defined to be 1. We can convert a formula with a product to a formula with a summation by using the identity

$$\lg \left(\prod_{k=1}^{n} a_k \right) = \sum_{k=1}^{n} \lg a_k .$$

Exercises

A.1-1
Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

A.1-2 ★
Show that $\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

A.1-3
Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

A.1-4 ★
Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

A.1-5 ★
Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

A.1-6
Prove that $\sum_{k=1}^{n} O(f_k(i)) = O\left(\sum_{k=1}^{n} f_k(i)\right)$ by using the linearity property of summations.

A.1-7
Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

A.1-8 ★
Evaluate the product $\prod_{k=2}^{n}(1 - 1/k^2)$.

A.2 Bounding summations

We have many techniques at our disposal for bounding the summations that describe the running times of algorithms. Here are some of the most frequently used methods.

Mathematical induction

The most basic way to evaluate a series is to use mathematical induction. As an example, let us prove that the arithmetic series $\sum_{k=1}^{n} k$ evaluates to $\frac{1}{2}n(n+1)$. We can easily verify this assertion for $n = 1$. We make the inductive assumption that it holds for n, and we prove that it holds for $n+1$. We have

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= \frac{1}{2}n(n+1) + (n+1) \\ &= \frac{1}{2}(n+1)(n+2) .\end{aligned}$$

You don't always need to guess the exact value of a summation in order to use mathematical induction. Instead, you can use induction to prove a bound on a summation. As an example, let us prove that the geometric series $\sum_{k=0}^{n} 3^k$ is $O(3^n)$. More specifically, let us prove that $\sum_{k=0}^{n} 3^k \le c3^n$ for some constant c. For the initial condition $n = 0$, we have $\sum_{k=0}^{0} 3^k = 1 \le c \cdot 1$ as long as $c \ge 1$. Assuming that the bound holds for n, let us prove that it holds for $n+1$. We have

$$\begin{aligned}\sum_{k=0}^{n+1} 3^k &= \sum_{k=0}^{n} 3^k + 3^{n+1} \\ &\le c3^n + 3^{n+1} \qquad \text{(by the inductive hypothesis)} \\ &= \left(\frac{1}{3} + \frac{1}{c}\right) c3^{n+1} \\ &\le c3^{n+1}\end{aligned}$$

as long as $(1/3 + 1/c) \le 1$ or, equivalently, $c \ge 3/2$. Thus, $\sum_{k=0}^{n} 3^k = O(3^n)$, as we wished to show.

We have to be careful when we use asymptotic notation to prove bounds by induction. Consider the following fallacious proof that $\sum_{k=1}^{n} k = O(n)$. Certainly, $\sum_{k=1}^{1} k = O(1)$. Assuming that the bound holds for n, we now prove it for $n+1$:

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= O(n) + (n+1) \qquad \Longleftarrow \textit{wrong!!} \\ &= O(n+1) .\end{aligned}$$

The bug in the argument is that the "constant" hidden by the "big-oh" grows with n and thus is not constant. We have not shown that the same constant works for *all* n.

Bounding the terms

We can sometimes obtain a good upper bound on a series by bounding each term of the series, and it often suffices to use the largest term to bound the others. For example, a quick upper bound on the arithmetic series (A.1) is

$$\begin{aligned}\sum_{k=1}^{n} k &\le \sum_{k=1}^{n} n \\ &= n^2 .\end{aligned}$$

In general, for a series $\sum_{k=1}^{n} a_k$, if we let $a_{\max} = \max_{1 \le k \le n} a_k$, then

$$\sum_{k=1}^{n} a_k \le n \cdot a_{\max} .$$

The technique of bounding each term in a series by the largest term is a weak method when the series can in fact be bounded by a geometric series. Given the series $\sum_{k=0}^{n} a_k$, suppose that $a_{k+1}/a_k \le r$ for all $k \ge 0$, where $0 < r < 1$ is a constant. We can bound the sum by an infinite decreasing geometric series, since $a_k \le a_0 r^k$, and thus

$$\begin{aligned}\sum_{k=0}^{n} a_k &\le \sum_{k=0}^{\infty} a_0 r^k \\ &= a_0 \sum_{k=0}^{\infty} r^k \\ &= a_0 \frac{1}{1-r} .\end{aligned}$$

We can apply this method to bound the summation $\sum_{k=1}^{\infty}(k/3^k)$. In order to start the summation at $k = 0$, we rewrite it as $\sum_{k=0}^{\infty}((k+1)/3^{k+1})$. The first term ($a_0$) is $1/3$, and the ratio (r) of consecutive terms is

$$\begin{aligned}\frac{(k+2)/3^{k+2}}{(k+1)/3^{k+1}} &= \frac{1}{3} \cdot \frac{k+2}{k+1} \\ &\le \frac{2}{3}\end{aligned}$$

for all $k \ge 0$. Thus, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{k}{3^k} &= \sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}} \\ &\le \frac{1}{3} \cdot \frac{1}{1-2/3} \\ &= 1 .\end{aligned}$$

A common bug in applying this method is to show that the ratio of consecutive terms is less than 1 and then to assume that the summation is bounded by a geometric series. An example is the infinite harmonic series, which diverges since

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} \\ &= \lim_{n\to\infty} \Theta(\lg n) \\ &= \infty .\end{aligned}$$

The ratio of the $(k+1)$st and kth terms in this series is $k/(k+1) < 1$, but the series is not bounded by a decreasing geometric series. To bound a series by a geometric series, we must show that there is an $r < 1$, which is a *constant*, such that the ratio of all pairs of consecutive terms never exceeds r. In the harmonic series, no such r exists because the ratio becomes arbitrarily close to 1.

Splitting summations

One way to obtain bounds on a difficult summation is to express the series as the sum of two or more series by partitioning the range of the index and then to bound each of the resulting series. For example, suppose we try to find a lower bound on the arithmetic series $\sum_{k=1}^{n} k$, which we have already seen has an upper bound of n^2. We might attempt to bound each term in the summation by the smallest term, but since that term is 1, we get a lower bound of n for the summation—far off from our upper bound of n^2.

We can obtain a better lower bound by first splitting the summation. Assume for convenience that n is even. We have

$$\begin{aligned}\sum_{k=1}^{n} k &= \sum_{k=1}^{n/2} k + \sum_{k=n/2+1}^{n} k \\ &\geq \sum_{k=1}^{n/2} 0 + \sum_{k=n/2+1}^{n} (n/2) \\ &= (n/2)^2 \\ &= \Omega(n^2) ,\end{aligned}$$

which is an asymptotically tight bound, since $\sum_{k=1}^{n} k = O(n^2)$.

For a summation arising from the analysis of an algorithm, we can often split the summation and ignore a constant number of the initial terms. Generally, this technique applies when each term a_k in a summation $\sum_{k=0}^{n} a_k$ is independent of n.

Then for any constant $k_0 > 0$, we can write

$$\begin{aligned}\sum_{k=0}^{n} a_k &= \sum_{k=0}^{k_0-1} a_k + \sum_{k=k_0}^{n} a_k \\ &= \Theta(1) + \sum_{k=k_0}^{n} a_k ,\end{aligned}$$

since the initial terms of the summation are all constant and there are a constant number of them. We can then use other methods to bound $\sum_{k=k_0}^{n} a_k$. This technique applies to infinite summations as well. For example, to find an asymptotic upper bound on

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k} ,$$

we observe that the ratio of consecutive terms is

$$\begin{aligned}\frac{(k+1)^2/2^{k+1}}{k^2/2^k} &= \frac{(k+1)^2}{2k^2} \\ &\le \frac{8}{9}\end{aligned}$$

if $k \ge 3$. Thus, the summation can be split into

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{k^2}{2^k} &= \sum_{k=0}^{2} \frac{k^2}{2^k} + \sum_{k=3}^{\infty} \frac{k^2}{2^k} \\ &\le \sum_{k=0}^{2} \frac{k^2}{2^k} + \frac{9}{8} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k \\ &= O(1) ,\end{aligned}$$

since the first summation has a constant number of terms and the second summation is a decreasing geometric series.

The technique of splitting summations can help us determine asymptotic bounds in much more difficult situations. For example, we can obtain a bound of $O(\lg n)$ on the harmonic series (A.7):

$$H_n = \sum_{k=1}^{n} \frac{1}{k} .$$

We do so by splitting the range 1 to n into $\lfloor \lg n \rfloor + 1$ pieces and upper-bounding the contribution of each piece by 1. For $i = 0, 1, \ldots, \lfloor \lg n \rfloor$, the ith piece consists of the terms starting at $1/2^i$ and going up to but not including $1/2^{i+1}$. The last piece might contain terms not in the original harmonic series, and thus we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &\le \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i + j} \\
&\le \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i} \\
&= \sum_{i=0}^{\lfloor \lg n \rfloor} 1 \\
&\le \lg n + 1 . \qquad \text{(A.10)}
\end{aligned}$$

Approximation by integrals

When a summation has the form $\sum_{k=m}^{n} f(k)$, where $f(k)$ is a monotonically increasing function, we can approximate it by integrals:

$$\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx . \qquad \text{(A.11)}$$

Figure A.1 justifies this approximation. The summation is represented as the area of the rectangles in the figure, and the integral is the shaded region under the curve. When $f(k)$ is a monotonically decreasing function, we can use a similar method to provide the bounds

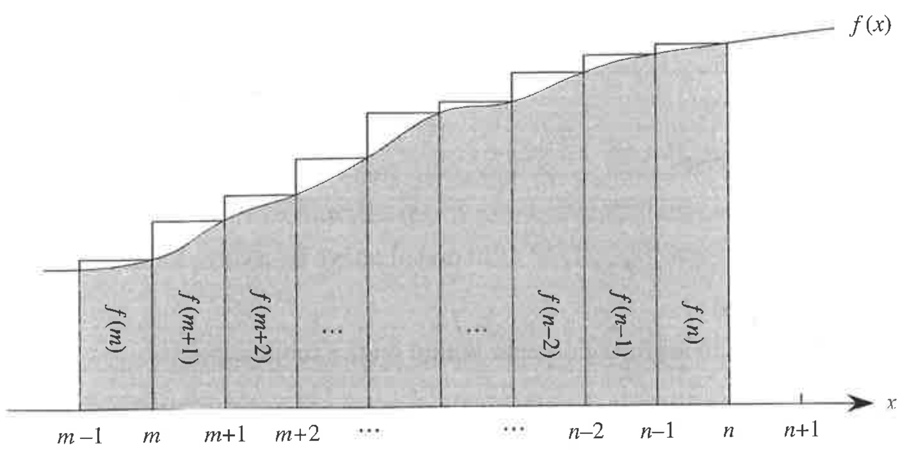
$$\int_{m}^{n+1} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)\,dx . \qquad \text{(A.12)}$$

The integral approximation (A.12) gives a tight estimate for the nth harmonic number. For a lower bound, we obtain

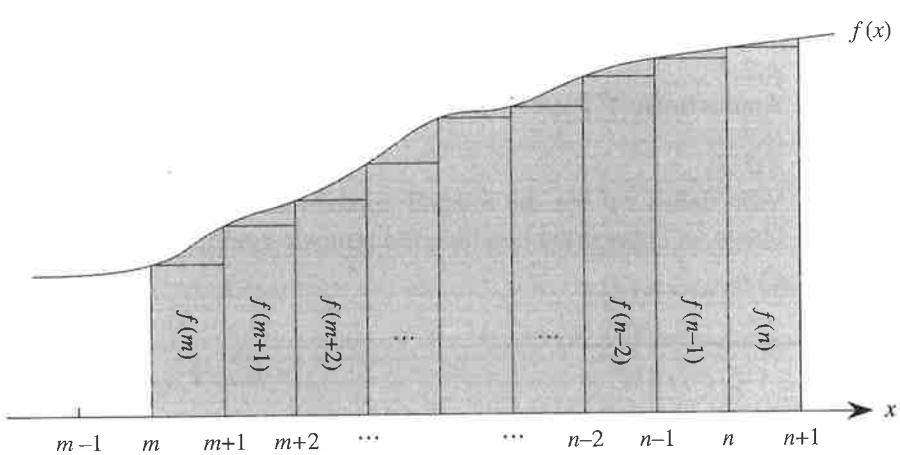
$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &\ge \int_{1}^{n+1} \frac{dx}{x} \\
&= \ln(n+1) . \qquad \text{(A.13)}
\end{aligned}$$

For the upper bound, we derive the inequality

$$\begin{aligned}
\sum_{k=2}^{n} \frac{1}{k} &\le \int_{1}^{n} \frac{dx}{x} \\
&= \ln n ,
\end{aligned}$$



(a)



(b)

Figure A.1 Approximation of $\sum_{k=m}^{n} f(k)$ by integrals. The area of each rectangle is shown within the rectangle, and the total rectangle area represents the value of the summation. The integral is represented by the shaded area under the curve. By comparing areas in **(a)**, we get $\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k)$, and then by shifting the rectangles one unit to the right, we get $\sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx$ in **(b)**.

which yields the bound

$$\sum_{k=1}^{n} \frac{1}{k} \le \ln n + 1 . \tag{A.14}$$

Exercises

A.2-1
Show that $\sum_{k=1}^{n} 1/k^2$ is bounded above by a constant.

A.2-2
Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil .$$

A.2-3
Show that the nth harmonic number is $\Omega(\lg n)$ by splitting the summation.

A.2-4
Approximate $\sum_{k=1}^{n} k^3$ with an integral.

A.2-5
Why didn't we use the integral approximation (A.12) directly on $\sum_{k=1}^{n} 1/k$ to obtain an upper bound on the nth harmonic number?

Problems

A-1 Bounding summations
Give asymptotically tight bounds on the following summations. Assume that $r \ge 0$ and $s \ge 0$ are constants.

a. $\displaystyle\sum_{k=1}^{n} k^r .$

b. $\displaystyle\sum_{k=1}^{n} \lg^s k .$

c. $\displaystyle\sum_{k=1}^{n} k^r \lg^s k$.

Appendix notes

Knuth [209] provides an excellent reference for the material presented here. You can find basic properties of series in any good calculus book, such as Apostol [18] or Thomas et al. [334].

B Sets, Etc.

Many chapters of this book touch on the elements of discrete mathematics. This appendix reviews more completely the notations, definitions, and elementary properties of sets, relations, functions, graphs, and trees. If you are already well versed in this material, you can probably just skim this chapter.

B.1 Sets

A ***set*** is a collection of distinguishable objects, called its ***members*** or ***elements***. If an object x is a member of a set S, we write $x \in S$ (read "x is a member of S" or, more briefly, "x is in S"). If x is not a member of S, we write $x \notin S$. We can describe a set by explicitly listing its members as a list inside braces. For example, we can define a set S to contain precisely the numbers 1, 2, and 3 by writing $S = \{1, 2, 3\}$. Since 2 is a member of the set S, we can write $2 \in S$, and since 4 is not a member, we have $4 \notin S$. A set cannot contain the same object more than once,[1] and its elements are not ordered. Two sets A and B are ***equal***, written $A = B$, if they contain the same elements. For example, $\{1, 2, 3, 1\} = \{1, 2, 3\} = \{3, 2, 1\}$.

We adopt special notations for frequently encountered sets:

- $\emptyset$ denotes the ***empty set***, that is, the set containing no members.
- $\mathbb{Z}$ denotes the set of ***integers***, that is, the set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{R}$ denotes the set of ***real numbers***.
- $\mathbb{N}$ denotes the set of ***natural numbers***, that is, the set $\{0, 1, 2, \ldots\}$.[2]

---

[1] A variation of a set, which can contain the same object more than once, is called a ***multiset***.

[2] Some authors start the natural numbers with 1 instead of 0. The modern trend seems to be to start with 0.

If all the elements of a set A are contained in a set B, that is, if $x \in A$ implies $x \in B$, then we write $A \subseteq B$ and say that A is a ***subset*** of B. A set A is a ***proper subset*** of B, written $A \subset B$, if $A \subseteq B$ but $A \neq B$. (Some authors use the symbol "$\subset$" to denote the ordinary subset relation, rather than the proper-subset relation.) For any set A, we have $A \subseteq A$. For two sets A and B, we have $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. For any three sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. For any set A, we have $\emptyset \subseteq A$.

We sometimes define sets in terms of other sets. Given a set A, we can define a set $B \subseteq A$ by stating a property that distinguishes the elements of B. For example, we can define the set of even integers by $\{x : x \in \mathbb{Z} \text{ and } x/2 \text{ is an integer}\}$. The colon in this notation is read "such that." (Some authors use a vertical bar in place of the colon.)

Given two sets A and B, we can also define new sets by applying ***set operations***:

- The ***intersection*** of sets A and B is the set

  $$A \cap B = \{x : x \in A \text{ and } x \in B\} .$$

- The ***union*** of sets A and B is the set

  $$A \cup B = \{x : x \in A \text{ or } x \in B\} .$$

- The ***difference*** between two sets A and B is the set

  $$A - B = \{x : x \in A \text{ and } x \notin B\} .$$

Set operations obey the following laws:

Empty set laws:

$$\begin{aligned} A \cap \emptyset &= \emptyset , \\ A \cup \emptyset &= A . \end{aligned}$$

Idempotency laws:

$$\begin{aligned} A \cap A &= A , \\ A \cup A &= A . \end{aligned}$$

Commutative laws:

$$\begin{aligned} A \cap B &= B \cap A , \\ A \cup B &= B \cup A . \end{aligned}$$

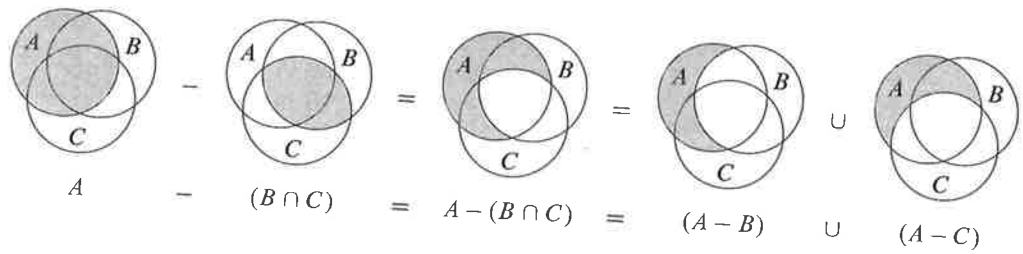


Figure B.1 A Venn diagram illustrating the first of DeMorgan's laws (B.2). Each of the sets A, B, and C is represented as a circle.

Associative laws:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C , \\ A \cup (B \cup C) &= (A \cup B) \cup C . \end{aligned}$$

Distributive laws:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) , \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) . \end{aligned} \tag{B.1}$$

Absorption laws:

$$\begin{aligned} A \cap (A \cup B) &= A , \\ A \cup (A \cap B) &= A . \end{aligned}$$

DeMorgan's laws:

$$\begin{aligned} A - (B \cap C) &= (A - B) \cup (A - C) , \\ A - (B \cup C) &= (A - B) \cap (A - C) . \end{aligned} \tag{B.2}$$

Figure B.1 illustrates the first of DeMorgan's laws, using a ***Venn diagram***: a graphical picture in which sets are represented as regions of the plane.

Often, all the sets under consideration are subsets of some larger set U called the ***universe***. For example, if we are considering various sets made up only of integers, the set $\mathbb{Z}$ of integers is an appropriate universe. Given a universe U, we define the ***complement*** of a set A as $\overline{A} = U - A = \{x : x \in U \text{ and } x \notin A\}$. For any set $A \subseteq U$, we have the following laws:

$$\begin{aligned} \overline{\overline{A}} &= A , \\ A \cap \overline{A} &= \emptyset , \\ A \cup \overline{A} &= U . \end{aligned}$$

We can rewrite DeMorgan's laws (B.2) with set complements. For any two sets $B, C \subseteq U$, we have

$$\begin{aligned}\overline{B \cap C} &= \overline{B} \cup \overline{C} \,, \\ \overline{B \cup C} &= \overline{B} \cap \overline{C} \,.\end{aligned}$$

Two sets A and B are ***disjoint*** if they have no elements in common, that is, if $A \cap B = \emptyset$. A collection $\mathcal{S} = \{S_i\}$ of nonempty sets forms a ***partition*** of a set S if

- the sets are ***pairwise disjoint***, that is, $S_i, S_j \in \mathcal{S}$ and $i \neq j$ imply $S_i \cap S_j = \emptyset$, and
- their union is S, that is,

$$S = \bigcup_{S_i \in \mathcal{S}} S_i \,.$$

In other words, $\mathcal{S}$ forms a partition of S if each element of S appears in exactly one $S_i \in \mathcal{S}$.

The number of elements in a set is the ***cardinality*** (or ***size***) of the set, denoted $|S|$. Two sets have the same cardinality if their elements can be put into a one-to-one correspondence. The cardinality of the empty set is $|\emptyset| = 0$. If the cardinality of a set is a natural number, we say the set is ***finite***; otherwise, it is ***infinite***. An infinite set that can be put into a one-to-one correspondence with the natural numbers $\mathbb{N}$ is ***countably infinite***; otherwise, it is ***uncountable***. For example, the integers $\mathbb{Z}$ are countable, but the reals $\mathbb{R}$ are uncountable.

For any two finite sets A and B, we have the identity

$$|A \cup B| = |A| + |B| - |A \cap B| \,, \tag{B.3}$$

from which we can conclude that

$$|A \cup B| \leq |A| + |B| \,.$$

If A and B are disjoint, then $|A \cap B| = 0$ and thus $|A \cup B| = |A| + |B|$. If $A \subseteq B$, then $|A| \leq |B|$.

A finite set of n elements is sometimes called an ***n-set***. A 1-set is called a ***singleton***. A subset of k elements of a set is sometimes called a ***k-subset***.

We denote the set of all subsets of a set S, including the empty set and S itself, by 2^S; we call 2^S the ***power set*** of S. For example, $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. The power set of a finite set S has cardinality $2^{|S|}$ (see Exercise B.1-5).

We sometimes care about setlike structures in which the elements are ordered. An ***ordered pair*** of two elements a and b is denoted (a, b) and is defined formally as the set $(a, b) = \{a, \{a, b\}\}$. Thus, the ordered pair (a, b) is *not* the same as the ordered pair (b, a).

The ***Cartesian product*** of two sets A and B, denoted $A \times B$, is the set of all ordered pairs such that the first element of the pair is an element of A and the second is an element of B. More formally,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\} .$$

For example, $\{a, b\} \times \{a, b, c\} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c)\}$. When A and B are finite sets, the cardinality of their Cartesian product is

$$|A \times B| = |A| \cdot |B| . \tag{B.4}$$

The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is the set of ***n-tuples***

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for } i = 1, 2, \ldots, n\} ,$$

whose cardinality is

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$

if all sets are finite. We denote an n-fold Cartesian product over a single set A by the set

$$A^n = A \times A \times \cdots \times A ,$$

whose cardinality is $|A^n| = |A|^n$ if A is finite. We can also view an n-tuple as a finite sequence of length n (see page 1166).

Exercises

B.1-1

Draw Venn diagrams that illustrate the first of the distributive laws (B.1).

B.1-2

Prove the generalization of DeMorgan's laws to any finite collection of sets:

$$\begin{aligned} \overline{A_1 \cap A_2 \cap \cdots \cap A_n} &= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} , \\ \overline{A_1 \cup A_2 \cup \cdots \cup A_n} &= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} . \end{aligned}$$

B.1-3 ★

Prove the generalization of equation (B.3), which is called the ***principle of inclusion and exclusion***:

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| = & \\
& |A_1| + |A_2| + \cdots + |A_n| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots \qquad \text{(all pairs)} \\
& + |A_1 \cap A_2 \cap A_3| + \cdots \qquad \text{(all triples)} \\
& \qquad \vdots \\
& + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| .
\end{aligned}
$$

B.1-4

Show that the set of odd natural numbers is countable.

B.1-5

Show that for any finite set S, the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

B.1-6

Give an inductive definition for an n-tuple by extending the set-theoretic definition for an ordered pair.

B.2 Relations

A ***binary relation*** R on two sets A and B is a subset of the Cartesian product $A \times B$. If $(a, b) \in R$, we sometimes write $a\ R\ b$. When we say that R is a binary relation on a set A, we mean that R is a subset of $A \times A$. For example, the "less than" relation on the natural numbers is the set $\{(a, b) : a, b \in \mathbb{N} \text{ and } a < b\}$. An n-ary relation on sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

A binary relation $R \subseteq A \times A$ is ***reflexive*** if

$$a\ R\ a$$

for all $a \in A$. For example, "=" and "≤" are reflexive relations on $\mathbb{N}$, but "<" is not. The relation R is ***symmetric*** if

$$a\ R\ b \text{ implies } b\ R\ a$$

for all $a, b \in A$. For example, "=" is symmetric, but "<" and "≤" are not. The relation R is ***transitive*** if

$$a\ R\ b \text{ and } b\ R\ c \text{ imply } a\ R\ c$$

for all $a, b, c \in A$. For example, the relations "$<$," "$\leq$," and "$=$" are transitive, but the relation $R = \{(a, b) : a, b \in \mathbb{N} \text{ and } a = b - 1\}$ is not, since $3 \; R \; 4$ and $4 \; R \; 5$ do not imply $3 \; R \; 5$.

A relation that is reflexive, symmetric, and transitive is an ***equivalence relation***. For example, "$=$" is an equivalence relation on the natural numbers, but "$<$" is not. If R is an equivalence relation on a set A, then for $a \in A$, the ***equivalence class*** of a is the set $[a] = \{b \in A : a \; R \; b\}$, that is, the set of all elements equivalent to a. For example, if we define $R = \{(a, b) : a, b \in \mathbb{N} \text{ and } a + b \text{ is an even number}\}$, then R is an equivalence relation, since $a + a$ is even (reflexive), $a + b$ is even implies $b + a$ is even (symmetric), and $a + b$ is even and $b + c$ is even imply $a + c$ is even (transitive). The equivalence class of 4 is $[4] = \{0, 2, 4, 6, \ldots\}$, and the equivalence class of 3 is $[3] = \{1, 3, 5, 7, \ldots\}$. A basic theorem of equivalence classes is the following.

Theorem B.1 (An equivalence relation is the same as a partition)
The equivalence classes of any equivalence relation R on a set A form a partition of A, and any partition of A determines an equivalence relation on A for which the sets in the partition are the equivalence classes.

Proof For the first part of the proof, we must show that the equivalence classes of R are nonempty, pairwise-disjoint sets whose union is A. Because R is reflexive, $a \in [a]$, and so the equivalence classes are nonempty; moreover, since every element $a \in A$ belongs to the equivalence class $[a]$, the union of the equivalence classes is A. It remains to show that the equivalence classes are pairwise disjoint, that is, if two equivalence classes $[a]$ and $[b]$ have an element c in common, then they are in fact the same set. Suppose that $a \; R \; c$ and $b \; R \; c$. By symmetry, $c \; R \; b$, and by transitivity, $a \; R \; b$. Thus, for any arbitrary element $x \in [a]$, we have $x \; R \; a$ and, by transitivity, $x \; R \; b$, and thus $[a] \subseteq [b]$. Similarly, $[b] \subseteq [a]$, and thus $[a] = [b]$.

For the second part of the proof, let $\mathcal{A} = \{A_i\}$ be a partition of A, and define $R = \{(a, b) : \text{there exists } i \text{ such that } a \in A_i \text{ and } b \in A_i\}$. We claim that R is an equivalence relation on A. Reflexivity holds, since $a \in A_i$ implies $a \; R \; a$. Symmetry holds, because if $a \; R \; b$, then a and b are in the same set A_i, and hence $b \; R \; a$. If $a \; R \; b$ and $b \; R \; c$, then all three elements are in the same set A_i, and thus $a \; R \; c$ and transitivity holds. To see that the sets in the partition are the equivalence classes of R, observe that if $a \in A_i$, then $x \in [a]$ implies $x \in A_i$, and $x \in A_i$ implies $x \in [a]$. ■

A binary relation R on a set A is ***antisymmetric*** if

$a \; R \; b$ and $b \; R \; a$ imply $a = b$.

For example, the "$\leq$" relation on the natural numbers is antisymmetric, since $a \leq b$ and $b \leq a$ imply $a = b$. A relation that is reflexive, antisymmetric, and transitive is a ***partial order***, and we call a set on which a partial order is defined a ***partially ordered set***. For example, the relation "is a descendant of" is a partial order on the set of all people (if we view individuals as being their own descendants).

In a partially ordered set A, there may be no single "maximum" element a such that $b \; R \; a$ for all $b \in A$. Instead, the set may contain several ***maximal*** elements a such that for no $b \in A$, where $b \neq a$, is it the case that $a \; R \; b$. For example, a collection of different-sized boxes may contain several maximal boxes that don't fit inside any other box, yet it has no single "maximum" box into which any other box will fit.[3]

A relation R on a set A is a ***total relation*** if for all $a, b \in A$, we have $a \; R \; b$ or $b \; R \; a$ (or both), that is, if every pairing of elements of A is related by R. A partial order that is also a total relation is a ***total order*** or ***linear order***. For example, the relation "$\leq$" is a total order on the natural numbers, but the "is a descendant of" relation is not a total order on the set of all people, since there are individuals neither of whom is descended from the other. A total relation that is transitive, but not necessarily reflexive and antisymmetric, is a ***total preorder***.

Exercises

B.2-1

Prove that the subset relation "$\subseteq$" on all subsets of $\mathbb{Z}$ is a partial order but not a total order.

B.2-2

Show that for any positive integer n, the relation "equivalent modulo n" is an equivalence relation on the integers. (We say that $a \equiv b \pmod{n}$ if there exists an integer q such that $a - b = qn$.) Into what equivalence classes does this relation partition the integers?

B.2-3

Give examples of relations that are

a. reflexive and symmetric but not transitive,

b. reflexive and transitive but not symmetric,

c. symmetric and transitive but not reflexive.

---

[3] To be precise, in order for the "fit inside" relation to be a partial order, we need to view a box as fitting inside itself.