Reading Assignments for Week 13

- Monday, April 14: Section 13.2 up to 13.2.4 (pp. 515–529), but you can skip the Tricks Behind the Math on pp. 522–524.
- Wednesday, April 16: Section 13.3 up to the middle of p. 534 (pp. 532–534) and Section 13.1 (pp. 509–515)
- Friday, April 18: No new reading. Catchup and review.

Homework #10 — due Friday, April 18

From lecture of Friday, April 11

1. **Problem N: Vesicle Free Energy.** We're going to consider a variant on the vesicle pulling expirement. Consider a vesicle which has been pulled with a force f and stretched into a very long spherocylinder, with length L much greater than the radius R of the cylinder and the two hemispherical caps. Determine the pulling force in terms of the surface tension τ and the bending modulus K_b.

2. **Problem O: Membrane Inclusion Energy.** The purpose of this problem is to derive the final result for the membrane inclusion free energy per unit length, Eq. (11.58), which can be written as

$$G_h(\text{per unit length}) = \frac{K_t^{3/4} K_b^{1/4}}{\sqrt{2} w_0^{3/2}} U^2. \qquad (\dagger)$$

The starting point is the free energy functional given in Eq. (11.36). The function $u(x)$ that minimizes this energy satisfies the Euler-Lagrange equation,

$$\frac{d^4u}{dx^4} + \frac{K_t}{K_b w_0^2} u = 0, \qquad (\ddagger)$$

with boundary conditions $u(0) = U$ and $u'(0) = 0$, and both u and u' vanishing as $x \to \infty$. Note: we're putting the boundary at $x = 0$, not $x = R$ as the book does.

(a) Show that $u(x) = Ue^{-kx}[\cos(kx) + \sin(kx)]$ (with real k) solves the differential equation ($\ddagger$) by plugging in the solution and determining the value of k.

(b) Show that this guess for $u(x)$ also satisfies the boundary conditions.

(c) Now that we have the function $u(x)$ that minimizes Eq. (11.36), we just need to plug it in and evaluate. Here is a trick that makes it, well, not too bad. Focus on the second integral in Eq. (11.36), and note that using Eq. ($\ddagger$) we can write

$$u(x)^2 = -\frac{K_b w_0^2}{K_t} u(x) \frac{d^4u}{dx^4}.$$

Use integration by parts (twice) and be careful with the boundary terms, to show that Eq. (11.36) becomes

$$G_h[u(x)] = \frac{K_b}{2} u(0) u'''(0).$$

That is, the both integrals will cancel out and won't need to be evaluated.

(d) Now evaluate the expression in (c) to show that it gives the expected free energy per length, Eq. (†).

From lecture of Monday, April 14

3. **Problem 13.4** *Note:* plot $c(t)/c_0$. That way you don't need a value for c_0. Take $a = L/2$.

4. **Problem P: FRAP On the Left:** The diffusion equation has solution Eq. (13.39) in the interval $-L \leq x \leq L$ when the concentration $c(x)$ is an even function. For an odd function, an expansion in sines is required.

   (a) Show that the expansion

   $$c(x,t) = \frac{c_0}{2} + \sum_{n=0}^{\infty} A_n(t) \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

   satisfies the boundary condition $\partial c/\partial x = 0$ at $x = \pm L$.

   (b) Plug this into the diffusion equation and derive an equation for the time evolution of the coefficients $A_n(t)$.

   (c) Consider an initial condition where the photobleaching has eliminated the fluoresence on the left half of the cell, i.e.

   $$c(x,0) = \begin{cases} 0 & \text{for } -L < x < 0 \\ c_0 & \text{for } 0 < x < L \end{cases}$$

   Find the initial coefficients $A_n(0)$ by using the orthogonality of the sine function:

   $$\int_{-L}^{L} \sin\left(\frac{(2n+1)\pi x}{2L}\right) \sin\left(\frac{(2m+1)\pi x}{2L}\right) dx = L\delta_{n,m}$$

   (d) Find the general solution for $c(x,t)$, analogous to Eq. (13.47) but for this odd (rather than even) initial condition.

From lecture of Wednesday, April 16

5. **Problem Q: Typical Diffusion Constants in Water.** Use the Stokes-Einstein relation, Eq. (13.62), to estimate the diffusion constant in water for various objects listed below. The viscosity of water is $\eta = 0.001\,\text{Pa·s}$.

   (a) an O_2 molecule

   (b) a large ion, which has an effective radius of 0.2 nm (because it drags some water around with it).

   (c) a typical protein

   (d) a yeast cell

6. **Problem R: Bacteria Size.** The size of a bacterium is limited by its ability to absorb nutrients, in particular O_2. To estimate this effect, assume the bacteria are spherical with radius R with the same mass density ρ as water. Assume for their cellular processes they consume oxygen at a rate r (in units of moles of O_2 per second per kilogram of bacteria).

(a) Find an expression for the maximum size of the bacterium in terms of the symbols above, the diffusion constant D of O_2 in water, and the concentration c_0 of O_2 in water.

(b) Using the values $r = 0.2\,\frac{\text{moles}}{\text{kg}\cdot\text{s}}$ for the consumption rate, $D = 2 \times 10^{-9}\,\text{m}^2/\text{s}$, and oxygen concentration $c_0 = 0.2\,\text{moles}/\text{m}^3$, find a numerical value for the upper limit on a bacteria size. Compare to the size of *E. coli*.