Reading Assignments for Week 4

- Monday, February 3: Sections 5.4 (pp. 214–219) and 5.7 (pp. 232–233)
- Wednesday, February 5: Section 5.5 (pp. 219–220) up to the paragraph beginning "As a concrete example...", then Sections 5.5.1 through Section 5.6 (pp. 222–232).
- Friday, February 7: No new reading. Catchup and review.

Homework #3 — due Friday, February 7

1. **Problem B: Chemical Equilibrium.** For the reaction $A \underset{k_2}{\overset{k_1}{\rightleftarrows}} B \overset{k_3}{\to} C$ we have the rate equations

$$\frac{da}{dt} = -k_1 a + k_2 b \qquad \frac{db}{dt} = k_1 a - k_2 b - k_3 b \qquad \frac{dc}{dt} = k_3 b$$

where, e.g., $a = [A]$ is the concentration of A molecules.

(a) Show that the total concentration $a + b + c$ is conserved, and that the equilibrium concentration of A molecules is $a_{\mathrm{eq}} = (k_2/k_1)b_{\mathrm{eq}}$.

(b) Write a program using Python or Mathematica or a spreadsheet to solve these equations and graph the solution as a function of time. Use Euler's method, with

$$a(t+\Delta t) \simeq a(t) + \Delta t \times \left.\frac{da}{dt}\right|_t = a(t) + \Delta t\Big(-k_1 a(t) + k_2 b(t)\Big)$$

and similar for $b(t+\Delta t)$ and $c(t+\Delta t)$. Take $k_1 = k_2 = 1$ and $\Delta t = 0.01$, and initial conditions $a_0 = 1$, $b_0 = c_0 = 0$.

Vary the value of k_3 to find (i) a case where $a(t)$ reaches equilibrium and (ii) another case where $a(t)$ does not reach equilibrium, similar to Fig. 5.6. Show the graphs and report the corresponding values of k_3.

2. **Problem C: Mechanical Equilibrium.** This is a problem illustrating the role of time scales in determining when we can assume mechanical equilibrium. Take a bead in an optical trap with spring constant k_{sp} subject to a time-dependent applied force $F_{\mathrm{app}}(t) = F_0 e^{-t/\tau}$, which describes gradually releasing the applied force.

(a) Using the equilibrium assumption, find $x_{\mathrm{eq}}(t)$.

(b) Now let's solve for the actual motion. The bead is in a viscous medium, so there is a friction force $F_{\mathrm{fr}} = -bv$. Newton's 2nd law gives us

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - k_{\mathrm{sp}}x + F_{\mathrm{app}}(t)$$

Let's assume the viscosity is high enough that the acceleration term is negligible, so the left hand side is zero. Solve this equation to find the resulting motion $x(t)$. Take as initial condition that the particle is in equilibrium with the applied force. *Hint* →

Hint: to solve the equation either (i) use the method of integration factors or (ii) guess the form $x(t) = Ae^{-t/\tau} + Be^{-(k_{\rm sp}/b)t}$ and determine A and B. If you're shaky on these methods, please come see me!

(c) Comparing time scales: show that in the limit of τ much greater than $b/k_{\rm sp}$ (i.e. slowly varying applied force) your solution for $x(t)$ reduces to the equilibrium assumption $x_{\rm eq}(t)$.

3. **Problem D: Stretched Beam.** Consider a beam of length L, such as shown in Fig. 5.23, that has been slowly stretched by an amount ΔL. Let's find the local amount of streching $u(x)$ where x ranges from 0 to L, under the assumption that the strain energy given by Eq. (5.28) is minimized. Solve the appropriate Euler-Lagrange equation for $u(x)$.

4. **Problem E: Bent Beam.**Consider a beam of length L, such as shown in Fig. 5.18a but inverted, that is fixed at one end and bent upward by an distance y at the other end. The equilibrium vertical displacement $u(x)$ is given by the minimum of the bending energy

$$E_{\rm bend} = \frac{1}{2} C \int_0^L \left(\frac{d^2 u}{dx^2} \right)^2 dx$$

where C is some material constant.

(a) This functional has second derivatives! Extend the treatment of Section 5.7 to the case where the integrand $f(u, u', u'')$ can depend on second derivatives, and show that the resulting Euler-Lagrange equation is

$$\frac{d^2}{dx^2} \left(\frac{\partial f}{\partial u''} \right) - \frac{d}{dx} \left(\frac{\partial f}{\partial u'} \right) + \frac{\partial f}{\partial u} = 0$$

Hint: along the way you should have an η'' term and you will need to use integration by parts twice on it. Assume that $\eta'(0) = \eta'(L) = 0$.

(b) Solve this equation to get the minimum energy shape of the bent beam. You will need the following boundary conditions: $u(0) = 0$ and $u(L) = y$ for the heights at each end. Additionally: $u'(0) = 0$ (tangent is horizontal at fixed end) and $u''(L) = 0$ (free end isn't bent).

Note: This is a simple case of what is known as Euler-Bernoulli beam theory.

5. Problem 5.3b. Assume a carbon atom for the mass.

6. **Problem F: Ideal gas law from entropy.** An ideal gas molecule has a number of microstates proportional to the volume V of its container. Use this idea and the thermodynamic relation $p/T = (\partial S/\partial V)_{E,N}$ to derive the ideal gas law.

7. Problem 5.8